MORE LOWENHEIM-SKOLEM RESULTS FOR ADMISSIBLE SETS

BY

MARK NADEL

ABSTRACT

It is shown that if A is a countable, admissible set and $\phi \in \mathcal{L}_A$, then under certain conditions, ϕ has a model in A. In general, however, if T is a consistent theory of $\mathcal{L}_A$, Σ-definable on A, then there is an admissible set $B \supseteq A$, with the same ordinals as A, containing a model of T.

A *downward Lowenheim-Skolem theorem* is a result of the following sort: Every semantically consistent theory of a certain type having a prescribed degree of simplicity has a model of some corresponding degree of simplicity. In the original downward Lowenheim-Skolem theory, the notion of simplicity involved was simply cardinality. In this paper, we continue the program initiated in [8] of presenting refinements of the original downward Lowenheim-Skolem theorem in which the notion of simplicity employed is based on the concept of an admissible set.

Our notion of a downward Lowenheim-Skolem theorem is very much like the familiar notion of basis theorem in recursion theory. In fact, there is a certain parallel between some of the results presented below and basis theorems. In Section 1, the path from basis theorem to downward Lowenheim-Skolem theorem involves more than just a translation into model theoretic terms. In concentrating on the structures themselves, one is hampered by the fact that a given countable structure always has continuum many isomorphic copies. One is therefore forced to consider theories rather than models.

Stated as downward Lowenheim-Skolem theorems, and proved model theoretically, additional results become available to those insufficiently initiated into the

Received November 13, 1973

subtleties of recursion theory to appreciate a counterpart expressed in its terms. Furthermore, results expressed in terms of theories and their models are generally more useful to the model theorist, and tiresome coding processes can be avoided. On the other hand, we do not mean to criticize the recursion theoretic approach, nor to deny the added insight which may be available to those able to view results in such terms. However, such an approach is often unnecessary, and may prove impractical for those lacking the recursion theoretic background.

It is assumed that the reader is familiar with the basic notions of infinitary logic and admissible sets. Any necessary background materical can be found in [1], [3], [6], or [9]. The admissible sets may be allowed to contain urelements as in [3], but unlike [3], we assume that admissible sets contain ω, that is, satisfy the axiom of infinity. In Section 2 we begin with an admissible set A and obtain another admissible set $B \supseteq A$; we may assume that B contains no additional urelements. As the use of urelements will not affect the presentation of the material to follow, we omit any further allusions.

Though the reader is most likely already aware of the completeness theorem for $\mathscr{L}_{\omega_1\omega}$, we mention it explicity. There is a complete notion of provability $\vdash$ for $\mathscr{L}_{\omega_1\omega}$. That is, if T is a countable theory in $\mathscr{L}_{\omega_1\omega}$ and ϕ is a sentence of $\mathscr{L}_{\omega_1\omega}$, then $T \vdash \phi$ iff $T \models \phi$. Specifically, we need to know:

(*) If A is admissible, $\phi \in \mathscr{L}_A$, and T is a theory in $\mathscr{L}_A$, Σ-definable on A, then the relation $T \vdash \phi$ is Σ-definable on A.

1.

We assume the reader is familiar with the basic material on consistency properties as presented in [6]. We fix a language $\mathscr{L}$ and a countably infinite set C of constant symbols disjoint from the set of symbols of $\mathscr{L}$. We will always assume that $\mathscr{L}$ and C are elements of any admissible set A under consideration. We denote by $\mathscr{L}'$ the language obtained from $\mathscr{L}$ by adjoining the additional constant symbols in C. If we choose $C \subseteq \omega$, then if $\mathscr{L}_B$ is a fragment of $\mathscr{L}_{\infty\omega}$, $\mathscr{L}'_B$ will be a fragment of $\mathscr{L}'_{\infty\omega}$. Let T be a consistent theory in $\mathscr{L}_B$. We denote by $\mathscr{S}(\mathscr{L}_B, T)$ the set of all finite sets s of sentences of $\mathscr{L}'_B$ such that the sentences of s contain only finitely many distinct constants of C, and such that not $T \vdash \neg \bigwedge s$.

Of basic importance to the study of $\mathscr{L}_{\infty\omega}$ is the following well-known lemma.

LEMMA 1. *If $\mathscr{L}_B$ is a countable fragment and T is a consistent theory in $\mathscr{L}_B$, then $\mathscr{S}(\mathscr{L}_B, T)$ is a consistency property.*

Lemma 1 is combined with the so-called model existence theorem to obtain the completeness theorem for $\mathscr{L}_{\omega_1\omega}$. We will state a more effective version of the model existence theorem in the extended version. A set $\mathscr{S}$ will be a consistency property in the sense of an admissible set A provided that $\mathscr{S}$ is a consistency property, $\mathscr{S} \in A$, each $s \in \mathscr{S}$ is countable in the sense of A, and each sentence $\phi \in \cup \mathscr{S}$ is a sentence of $\mathscr{L}_{\omega_1\omega}$ in the sense of A; that is, each conjunction and disjunction in ϕ is over a set of formulas countable in the sense of A. In particular, if $\mathscr{S} \in A$ is a consistency property and there is some $\mathscr{L}_B \in A$, a countable fragment in the sense of A, such that each $s \in S$ is a subset of $\mathscr{L}_B$, then S is a consistency property in the sense of A. In this paper, consistency properties will always consist of finite sets of sentences. We can now state

LEMMA 2. *Suppose A is admissible and $\mathscr{S}$ is a consistency property in the sense of A. Further suppose that T is a set of sentences, countable in the sense of A, such that for each $s \in \mathscr{S}$ and $\phi \in T$, $s \cup \{\phi\} \in \mathscr{S}$. Then, for each $s \in \mathscr{S}$, $s \cup T$ has a model in A.*

The proof of this form of the model existence theorem involves nothing beyond noting that the usual proof proceeds sufficiently effectively.

Now, if $\mathscr{L}_B$ is a countable fragment of $\mathscr{L}_{\infty\omega}$ in the sense of the countable admissible set A, and T is a consistent theory in $\mathscr{L}_B$ which is Σ-definable on A, then, in view of (*), $\mathscr{S}(\mathscr{L}_B, T) \in A^+$, the smallest admissible set with A as an element. Combining this last fact with Lemmas 1 and 2 above we have the following result which was already implicit in [8].

THEOREM 1. *Let $\mathscr{L}_B$ be a countable fragment of $\mathscr{L}_{\infty\omega}$ in the sense of the admissible set A, and let T be a consistent theory in $\mathscr{L}_B$, Σ-definable on A. Then T has a model in A^+.*

We observe the following variant of Theorem 1: If A is countable in A^+, and T is a consistent theory in $\mathscr{L}_A$, Σ-definable on A, then T has a model in A^+.

We noted in [8] that one cannot improve Theorem 1 by changing A^+ to A. In fact, if A is not recursively inaccessible, a counterexample always exists. However, if we consider the special case in which T is a complete theory we can replace A^+ with A. A *complete theory* for $\mathscr{L}_B$ is, of course, a consistent theory T such that for each $\phi \in \mathscr{L}_B$, either $T \vdash \phi$ or $T \vdash \neg \phi$. Then, for T Σ on A and any $\phi \in \mathscr{L}_B$, the relation $T \vdash \phi$ is Δ on A, since we have $T \vdash \phi$ iff not $T \vdash \neg \phi$. We note that in this case, if $T^* = \{\phi \in \mathscr{L}_B : T \vdash \phi\}$, T^* is equivalent to T and $T^* \in A$. Hence we may limit our consideration to complete theories $T \in A$. Furthermore, even for

$\phi \in \mathscr{L}'_B$ the relation $T \vdash \phi$ is Δ on A, since $T \vdash \phi$ iff $T \vdash \psi$ where ψ is the sentence of $\mathscr{L}_B$ obtained from ϕ by replacing distinct constants of C appearing in ϕ by distinct variables not appearing in ϕ, and universally quantifying over these new variables. This last procedure is legitimate since sentences of $\mathscr{L}'_B$ contain only finitely many constants, and is effective.

As a consequence of the above we have $\mathscr{S}(\mathscr{L}_B, T) \in A$, and can now immediately obtain the following theorem.

THEOREM 2. *Let $\mathscr{L}_B$ be a countable fragment in the sense of the admissible set A, and let $T \in A$ be a complete theory in $\mathscr{L}_B$. Then T has a model in A.*

In particular we obtain

COROLLARY. *Let $\mathscr{L}_B$ be a countable fragment in the sense of the countable admissible set A, and let $T \in A$ be a theory in $\mathscr{L}_B$. If T is ω-categorical, then T has a model in A.*

We observe that a Δ-definable theory on a locally countable admissible set A need not have a model in A, even if the theory is complete for $\mathscr{L}_A$.

We will need the following special case of the omitting types theorem in the effective version.

THEOREM 3. *Let $\mathscr{L}_B$ be a countable fragment in the sense of the admissible set A, and let $T \in A$ be a complete theory in $\mathscr{L}_B$. Suppose $\Phi \in A$ is a function with domain ω, such that for each $n \in \omega$, $\Phi(n)$ is a set of formulas of $\mathscr{L}_B$ with at most the variables $x_0, \cdots, x_{i_n}$ occurring free. If, for each $\psi \in \mathscr{L}_B$ consistent with T, with free variables among $x_0, \cdots, x_{i_n}$, there is a $\phi \in \Phi(n)$ such that $T \cup \{\psi, \phi\}$ is consistent, then the theory*

$$T \bigcup \{(\forall x_0) \cdots (\forall x_{i_n}) \bigvee \Phi(n) : n \in \omega\}$$

has a model in A.

The proof of Theorem 3 can be taken from the standard proof of the omitting types theorem in [6], with the additional considerations as in the proof of Theorem 2 above.

We assume the reader is familiar with the notion of a prime model. The necessary information is available in [6, Chapter 12].

THEOREM 4. *Let $\mathscr{L}_B$ be a countable fragment in the sense of the admissible set A, and let $T \in A$ be a complete theory in $\mathscr{L}_B$. If T has a prime model, then there is a prime model of T in A.*

PROOF. A countable model $\mathfrak{M}$ of T will be a prime model for T provided that each n-tuple of elements of the model satisfies a formula of $\mathscr{L}_B$ complete with respect to T for formulas of $\mathscr{L}_B$. Let K_n be the set of formulas of $\mathscr{L}_B$ with free variables among $x_0, \cdots, x_{n-1}$, complete with respect to T for formulas of $\mathscr{L}_B$. It is straightforward to verify (see [9, Sect. 5]) that the function K with domain ω, such that $K(n) = K_n$ is in A. Now, we take $\Phi = K$ in Theorem 3 above. Since we assume T has a prime model, the hypothesis of Theorem 3 is satisfied, and so T has a model in A, every n-tuple of whose elements satisfies a formula of $\mathscr{L}_B$ complete for $\mathscr{L}_B$ with respect to T. Since this model is countable, it is a prime model for T. ∎

It is known (see [6]) that if a complete theory T in the countable fragment $\mathscr{L}_B$ admits only countably many types over $\mathscr{L}_B$, then T has a prime model. In particular, if T has fewer than $2^{\aleph_0}$ non-isomorphic countable models, then T admits only countably many types over $\mathscr{L}_B$, and so T has a prime model in A. We can obtain a stronger result in this direction, but without mention of prime models, using Lemma 3 below.

Given a consistent theory T in a fragment $\mathscr{L}_B$, by a *completion of T in $\mathscr{L}_B$* we mean a consistent theory $T' \supseteq T$ in $\mathscr{L}_B$, complete for $\mathscr{L}_B$ in the strong sense that for each $\phi \in \mathscr{L}_B$, either $\phi \in T'$ or $\neg\phi \in T'$. If T is complete for $\mathscr{L}_B$, T will have a unique completion in $\mathscr{L}_B$. When only one fragment $\mathscr{L}_B$ is involved, we omit mentioning the fragment each time.

LEMMA 3. *Let A be a countable admissible set, and $\mathscr{L}_B \in A$ a fragment. If a consistent theory T in $\mathscr{L}_B$, Σ-definable on A, has fewer than $2^{\aleph_0}$ completions, then every completion of T is in A.*

A direct proof of the above follows from [3, Chap. III, Sect. 8]. Restating the result there in the form we need, we have that if some consistent theory T' on $\mathscr{L}_A$ has fewer than $2^{\aleph_0}$ completions on $\mathscr{L}_B$, then $T' \cup \{\phi\}$ is complete for $\mathscr{L}_B$ for some $\phi \in \mathscr{L}_A$. Now let

$$T' = T \bigcup \{\neg \bigwedge T^*: T^* \in A \,\&\, (\forall \phi \in \mathscr{L}_B)\ [\phi \in T^* \leftrightarrow \neg\ \phi \notin T^*]\}.$$

If some completion of T on $\mathscr{L}_B$ is not in A, then T' is consistent. But T' has fewer than $2^{\aleph_0}$ completions on $\mathscr{L}_B$, so for some $\phi \in \mathscr{L}_A$, $T' \cup \{\phi\}$ is complete for $\mathscr{L}_B$. Now, we note that $T' \cup \{\phi\}$ is Σ-definable on A since T is. Hence $\{\psi \in \mathscr{L}_B: T' \cup \{\phi\} \vdash \psi\}$ is in A, and is a completion of T on $\mathscr{L}_B$. Then of course, $\neg \bigwedge \{\psi \in \mathscr{L}_B: T' \cup \{\phi\} \vdash \psi\} \in T'$, whence $T' \cup \{\phi\}$ is inconsistent, a contradiction. We now see that each completion of T on $\mathscr{L}_B$ is of the form $\{\psi \in \mathscr{L}_B : T^* \vdash \psi\}$, for some $T^* \in A$, and so, is in A.

Alternatively, Lemma 3 is seen to follow from the more classical result that if a Σ_1^1 subset S of reals has cardinality less than the continuum, S is a set of hyperarithmetic reals. In [3], an extended form of Lemma 3 is used to obtain this last result. Further generalizations of Lemma 3 are given in [3]. One generalization is to the case in which one considers $\mathscr{L}_A$ itself rather than some $\mathscr{L}_B \in A$. Though this result provides a theory Δ-definable on A, one cannot, in general, obtain a model in A from the theory. Another generalization in which one considers two languages will be mentioned later in this section.

Combining Lemma 3 with Theorem 2 we easily obtain

THEOREM 5. *Let $\mathscr{L}_B$ be a countable fragment in the sense of the countable admissible set A, and suppose T is a consistent theory in $\mathscr{L}_B$ which is Σ-definable on A, and has fewer than $2^{\aleph_0}$ completions. Then T has a model in A. In fact, every model of T has the same $\mathscr{L}_B$ theory as some model in A.*

Continuing in this direction we obtain a result whose hypothesis is weaker than that of Theorem 5. We assume the reader is familiar with the notion of the canonical Scott sentence of a model. We denote the canonical Scott sentence of a model $\mathfrak{M}$ by css $(\mathfrak{M})$. Basic information on css $(\mathfrak{M})$ is available in [9]. It is easy to verify that if $\mathscr{L}_B$ is a fragment in A, and $T \in A$ is a complete theory in $\mathscr{L}_B$, and $\mathfrak{M} \in A$ is a prime model for T, then css $(\mathfrak{M}) \in A$. We note that if A is an admissible set, and $T \in A$ is a theory in $\mathscr{L}_A$, then the class of all canonical Scott sentences in A of models of T in A is Σ-definable on A. We denote this class by Γ_T. Γ_T may be empty, either because T has no models in A, or because no model of T in A has its canonical Scott sentence in $\mathscr{L}_A$.

We now state the aforementioned result for the case in which A is locally countable. In this case, if $\mathfrak{M} \models T$, css $(\mathfrak{M}) \in A$ iff css $(\mathfrak{M}) \in \Gamma_T$. We comment on the general case thereafter.

THEOREM 6. *Let $\mathscr{L}_B$ be a fragment in the locally countable admissible set A and let $T \in A$ be a consistent theory in $\mathscr{L}_B$ which has fewer than $2^{\aleph_0}$ countable models up to isomorphism. Then either*

(i) *$\Gamma_T \in A$ and the canonical Scott sentence of each model of T is in Γ_T, or*

(ii) *$\Gamma_T \notin A$, Γ_T is infinite, and there is a model of T whose canonical Scott sentence is not in Γ_T.*

PROOF. Suppose (i) fails. Then either $\Gamma_T \notin A$, or $\Gamma_T \in A$ and there is a model $\mathfrak{M}$ of T such that css $(\mathfrak{M}) \notin \Gamma_T$.

In the first case we consider the theory $S = T \cup \{\neg\sigma : \sigma \in \Gamma_T\}$. Since both T and Γ_T are Σ-definable on A, S is also Σ-definable on A. Furthermore, $S \notin A$, or else, since $T \in A$, $\Gamma_T = S \setminus T$ would be an element of A by Δ-separation. One can now apply the Barwise compactness theorem to S, obtaining a model $\mathfrak{M}$ of T with css $(\mathfrak{M}) \notin \Gamma_T$.

In the second case, our assumption implies that S has a model. Since $T \in A$ and $\Gamma_T \in A$, there is a fragment $\mathscr{L}_{B'} \in A$ such that S is a theory in $\mathscr{L}_{B'}$. Since T has fewer than $2^{\aleph_0}$ countable models up to isomorphism, so does S. Finally, since A is locally countable, $\mathscr{L}_{B'}$ is a countable fragment in the sense of A, and so, by our earlier observations, there is a prime model $\mathfrak{M}$ for S in A. As we noted above, css $(\mathfrak{M}) \in A$. Then, of course, css $(\mathfrak{M}) \in \Gamma_T$, which is impossible. We conclude therefore that $\Gamma_T \notin A$, and we are back to the first case.

Since A is closed under finite subsets, it is obvious that if $\Gamma_T \notin A$, then Γ_T is infinite. ∎

If A is not locally countable, then Γ_T may be in A, but uncountable in the sense of A. In this case our proof would not yield the conclusion that the canonical Scott sentence of each model of T is in Γ_T.

If T were assumed to be only Σ-definable on A, then Γ_T need not be Σ-definable on A, and the Barwise compactness theorem could not be applied to S. However, one can still show that every model of T has a Scott sentence in Γ_T, or that, at least, Γ_T is infinite.

In this case, we first consider the $\mathscr{L}_B$ completions of T, which are all in A by Theorem 5. Now, by observations made above, each completion has a prime model in A, with its canonical Scott sentence in A. If there are infinitely many completions, we are done, as well as if every countable model of T is isomorphic to one of these prime models. Otherwise, we consider the theory $T_1 \in A$ obtained from T by adjoining the negations of these finitely many canonical Scott sentences. T_1 is again a consistent theory in some fragment $\mathscr{L}_B$, countable in the sense of A, with fewer than $2^{\aleph_0}$ non-isomorphic models, and we may proceed as before. Iterating this process, if there are only finitely many non-isomorphic countable models of T, a copy of each, along with the canonical Scott sentence will be obtained in A. Otherwise, infinitely many non-isomorphic models of T and their canonical Scott sentences will be generated in A.

Lemma 3, and hence Theorems 5 and 6 also hold in more general pseudo-elementary versions. For brevity, we state the pseudo-elementary version of Theorem 5 explicitly and omit the others.

THEOREM 5′. *Suppose $\mathscr{L}_B$ is a countable fragment in the sense of the countable admissible set A, $\mathscr{K}_D$ is a fragment in A, and $\mathscr{L}_B \subseteq \mathscr{K}_D$. Let T be a theory in $\mathscr{K}_D$ such that there are fewer than $2^{\aleph_0}$ completions in $\mathscr{L}$ consistent with T. Then every model of T has the same $\mathscr{L}_B$ theory as some model in A.*

One might wonder if, as long as we assume that A is countable, and so $\mathscr{L}_B \in A$ is also countable, we might be able to drop the assumption that $\mathscr{L}_B$ is countable in A, in some or all of the results above. However, this cannot be done for any of the above results, since one can find an example of a countable admissible A and a sentence $\phi \in \mathscr{L}_A$ which is ω-categorical and has no model in A. In fact, A can be taken to be recursively inaccessible. We briefly indicate below how such an example might be found.

In [4] Gregory adapts an example of Malitz [7] to obtain a sentence ϕ of $\mathscr{L}_{\infty\omega}$ which is syntactically complete for $\mathscr{L}_{\infty\omega}$ (with the proof system analogous to the usual system for $\mathscr{L}_{\omega_1\omega}$), but which has no model. If we let A be a countable transitive model of ZF or of that finite part of ZF necessary to carry out Gregory's argument, there will be a sentence $\phi \in \mathscr{L}_A$ which is ω-categorical, but has no model in A itself. The fact that ϕ is ω-categorical follows from the actual meaning of ϕ, which we do not discuss here (see [7]).

2.

Just as Theorems 5 and 6 were motivated by a more classical result of recursion theory, so was the principal result of this section. Our starting point in this case was the so-called Kleene-Gandy Basis Theorem, which states that every non-empty Σ_1^1 set of reals contains a real of strictly lower hyperdegree then Kleene's $\mathcal{O}$. While the object in Theorems 5 and 6 was to consider theories with few non-isomorphic countable models, the idea here is to look for models in fattenings o A. Though any result in this direction will have a weaker conclusion than any of our previous results, with the possible exception of Theorem 1, we will also be able to manage with an extremely weak hypothesis.

In recent years, a number of results have been obtained which may be regarded as concerning fattenings. One may view results dealing with admissible ordinals, rather than admissible sets, in terms of fattenings. We shall mention some of these below as corollaries. We now mention a very general result in this direction, which though no more difficult to prove than certain known results, appears to have been overlooked. This result will allow us to view several specialized results, whose relationship might appear somewhat remote, as instances of a single Lowenheim-Skolem phenomenon in infinitary logic.

THEOREM 7. *Suppose A is a countable admissible set and T is a consistent theory of $\mathscr{L}_A$, Σ-definable on A. Then, there is an admissible set $B \supseteq A$, having the same ordinals as A, which contains a model of T.*

Using results of [9] we obtain

COROLLARY 1. *Suppose A is a countable admissible set and T is a consistent theory of $\mathscr{L}_A$, Σ-definable on A. Then T has a model whose canonical Scott sentence has quantifier rank at most $\alpha + \omega$, where α is the least ordinal not in A.*

Our original proof of Theorem 7 involved an application of the Kleene-Gandy Basis Theorem, which, in turn, can easily be proved from Theorem 7. Subsequently we noticed that a proof very similar to that of [2, (6.1)] could be used. Rather than using $\mathscr{L}$ directly, we consider a theory T' in the language of set theory augmented by a constant symbol $\underline{a}$ for each element a of A, and an additional constant symbol $\mathfrak{M}$. As part of T', we include $\mathfrak{M} \models \phi$ for each $\phi \in T$. This is possible since $\mathscr{L} \in A$, and T is Σ-definable on A. Furthermore, we demand that *the universe of $\mathfrak{M}$ is ω* so that the interpretation of $\mathfrak{M}$ will be included in the standard part of the model of KP produced. The absoluteness of the notions involved presents no difficulty. We leave the exact details of the proof to the reader. This proof involves both the Omitting Types Theorem and the Barwise Compactness Theorem. Weaker version of Theorem 7 can be obtained from the Barwise Compactness Theorem alone. For example this is true if one considers only a single sentence of $\mathscr{L}_A$ or even a Σ-definable theory whose sentences are bounded in rank in A, or if one assumes that A is a successor admissible. The reader may be familiar with the omitting types proof of Grilliot and Simpson for [6, Th. 15]. Their proof can easily be adapted to cover the cases where A is of the form $L_\alpha(a)$, or where $L_{0(A)}(a)$ is projectible for some $a \in A$. However, while some knowledge of the recursion theoretic aspects of admissible sets is required for that proof, none is required in the proof in [2].

COROLLARY 2. (*Sacks* [10]). *For each countable admissible ordinal α there is a set $x \subseteq \omega$ such that $\omega_1^x = \alpha$.*

PROOF. We may begin with the admissible set $A = L_\alpha$. For $\mathscr{L}$ we choose the language of set theory with an additional constant c. By a Σ recursion in A, one can easily define for each ordinal $\beta \in A$ a formula $\phi_\beta(x)$ which characterizes the order type of β. For the Σ-definable theory T we take a collection of sentences of $\mathscr{L}_A$ expressing the following:

(i) axioms of KP,

(ii) $c \subseteq \omega$

(iii) $(\exists x)\phi_\beta(x)$ and x is recursive in c for each ordinal $\beta \in A$.

By Theorem 7 we find a model $\mathfrak{M}$ of T in an admissible set B with ordinal α. Let A^* be the standard part of $\mathfrak{M}$. A^* is easily seen to have ordinal α, and if x is the interpretation of c in $\mathfrak{M}$, then x is in A^*, and hence $\omega_1^x \leqq \alpha$. On the other hand, $\omega_1^x \geqq \alpha$ because of (iii). ∎

In the above, the general model theory has been taken care of in Theorem 7, and the proof need only touch on the specific considerations involved, for example, the absoluteness of the relations employed.

We leave the easy proof of the next application to the reader.

COROLLARY 3. (Friedman.) *Let A be a countable admissible set and let T be a theory of $\mathscr{L}_A$ extending KP, which is Σ-definable on A. If T has a model which is an end extension of A, then T has a model which is an end extension of A and whose standard ordinals are just the ordinals of A.*

The following is merely a special case of Corollary 3, but is often more useful in practice. A theory T extending KP is said to have the *truncation property* iff the standard part of every model of T is a model of T. We have already implicitly used the fact that KP itself has the truncation property in the proof of Corollary 2, and in the proof of Theorem 7 itself.

COROLLARY 4. *Let A be a countable admissible set and let T be a theory in $\mathscr{L}_A$ extending KP, which has the truncation property and which is Σ-definable on A If A has an end extension satisfying T, then there is an admissible set $B \supseteq A$, with the same ordinals as A, which is the universe of a model of T.*

As simple applications of Corollary 4, given a countable admissible set A, we may find an extension of A to an admissible set with the same ordinals in which every set is countable, or in which the power set axiom holds, or in which every ordinal is countable, but not every set. An important drawback regarding the use of Corollary 4 is that ZF does not have the truncation property. Nonetheless, Corollary 4 is still very strong. In particular, Theorem 7 itself could have been derived from it in a fairly direct way. Our reason for listing one result as theorem and the other as corollary is simply a matter of our feeling as to which statement appears more general.

As a final corollary to Theorem 7 we state a strengthening which follows easily by a familiar trick. This result, which we will state as Corollary 5 below, may be regarded as a generalization, in the language of admissible sets, of the Kleene-Gandy Basis Theorem.

For the purposes of the discussion to follow we fix two alphabets $\mathscr{L}$, $\mathscr{L}' \in A$, with $\mathscr{L} \subseteq \mathscr{L}'$. Following [3], we say that ϕ is a Σ_1^1 formula of $\mathscr{L}_A$ iff there is a formula ψ of $\mathscr{L}'_A$ such that ϕ is of the form $(\exists Q)\psi$, where $Q \in A$ is a set of symbols of $\mathscr{L}' \setminus \mathscr{L}$. Given a Σ_1^1 formula of $\mathscr{L}_A$, $\phi = (\exists Q)\psi$, we denote by ϕ^* the formula ψ of $\mathscr{L}'_A$.

We fix a particular symbol $\underset{\sim}{R}$ of $\mathscr{L}' \setminus \mathscr{L}$. We say that a Σ_1^1-formula $\phi = (\exists Q)\psi$ of $\mathscr{L}_A$ is an $\underset{\sim}{R}$-formula iff the only symbol of $\mathscr{L}' \setminus \mathscr{L}$ occurring in ψ but not in Q is $\underset{\sim}{R}$. For simplicity we restrict our discussion to $\underset{\sim}{R}$-sentences, which are simply $\underset{\sim}{R}$-formulas $(\exists Q)\psi$ in which ψ is a sentence of $\mathscr{L}'_A$. We leave it to the reader to extend the discussion to arbitrary Σ_1^1-formulas of $\mathscr{L}_A$. By an $\underset{\sim}{R}$-theory we simply mean a set of $\underset{\sim}{R}$-sentences.

If T is an $\underset{\sim}{R}$-theory, and $\mathfrak{M}$ is a structure for $\mathscr{L}$, we say that R *satisfies* T *in* $\mathfrak{M}$ iff there is an expansion of $\mathfrak{M}$ to a structure $\mathfrak{M}'$ for $\mathscr{L}'$, in which $\underset{\sim}{R}$ is interpreted by R, and such that $\mathfrak{M}' \models \phi^*$ for each $\phi \in T$.

The result we are about to state concerning $\underset{\sim}{R}$-theories is perhaps most interesting when we begin with a fixed structure as well as an R-theory.

Corollary 5. *Let A be a countable admissible set, $\mathscr{L}$, $\mathscr{L}' \in A$ alphabets, with $\mathscr{L}' \supseteq \mathscr{L}$, and $\underset{\sim}{R} \in \mathscr{L}' \setminus \mathscr{L}$. Let $\mathfrak{M} \in A$ be a structure for $\mathscr{L}$. Let T be an R-theory, Σ-definable over A, such that some R satisfies T in $\mathfrak{M}$. Then, there is an admissible set $B \supseteq A$ with the same ordinals as A, and an $R \in B$, such that R satisfies T in $\mathfrak{M}$.*

Proof. We will work in an alphabet $\mathscr{L}^* \in A$ which is obtained from $\mathscr{L}'$ by adding a new constant symbol c_m, for each $m \in M$. We consider the following theory T^* of $\mathscr{L}_A^*$ which is easily seen to be Σ-definable on A:

(i) atomic diagram of $\mathfrak{M}$,

(ii) $(\forall x) \bigvee_{m \in M} x = c_m$,

(iii) ϕ^* for each $\phi \in T$.

By our hypothesis on T, T^* has a model. By Theorem 7, here is an admissible set $B \supseteq A$ with the same ordinals as A, and a model $\mathfrak{M}^*$ of T^* in B. Clearly, $\mathfrak{M}^*$ is isomorphic in B to a model $\mathfrak{N}$, in which the interpretation of each c_m is m itself. Now, we need merely choose R to be the interpretation of $\underline{R}$ in $\mathfrak{N}$. ∎

The reader will have probably already noticed that the notion of pseudo-elementary class as used in Section 1 and that of Σ_1^1 theory are really equivalent. Our choice of nomenclature was influenced by common usage.

More sophisticated recursion theoretic results in this direction may be obtained. We recommend that the reader consult [5].

References

1. J. Barwise, *Infinitary logic and admissible sets*, Doctoral Dissertation, Standford University, 1967.

2. J. Barwise, *Admissible sets over models of set theory*, to appear.

3. J. Barwise, *Admissible sets*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, to appear.

4. J. Gregory, *Incompleteness of a formal system for infinitary finite-quantifier formulas*, J. Symbolic Logic **36** (1971), 445–455.

5. T. J. Grilliot, *Omitting types; application to recursion theory*, J. Symbolic Logic **37** (1972), 81–89.

6. H. J. Keisler, *Model theory for infinitary logic*, North Holland, Amsterdam, 1971.

7. J. Maltiz, *On the Hanf number of complete $L_{\omega_1\omega}$ sentences*, Lecture Notes in Mathematics 72, Springer-Verlag, Berlin, 1968, 166–181.

8. M. Nadel, *Some Lowenheim-Skolem results for admissible sets*, Israel J. Math. **12** (1972), 427–432.

9. M. Nadel, *Scott sentences and admissible sets*, Anal. of Math. Logic to appear.

10. G. E. Sacks, *Countable admissible ordinals and hyperdegrees*, to appear.

Department of Mathematics
California Institute of Technology
Pasadena, California, U. S. A.